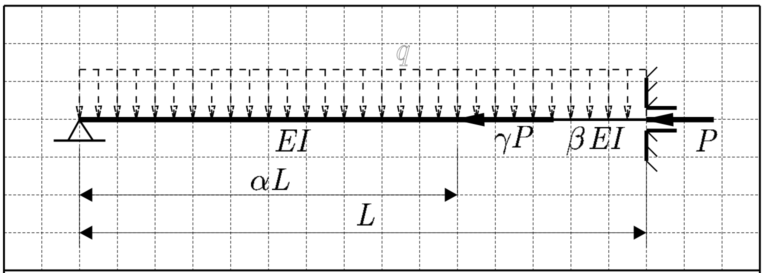
Przykład do zadania 6.

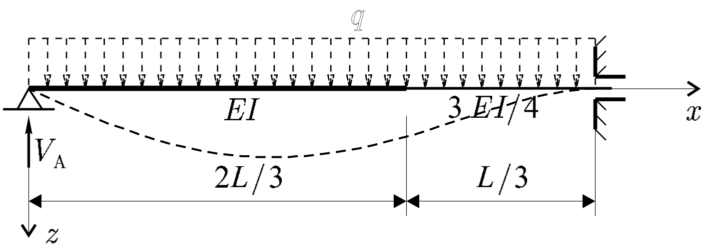
Dla pręta pokazanego na rysunku wyznaczyć krytyczną wartość siły P oraz współczynnik wyboczeniowy μ. Wykorzystać energetyczne kryterium Timoshenki przyjmując jako postulowaną postać wyboczenia linię ugięcia belki wyznaczoną dla zadanego obciążenia rozłożonego q.



$\alpha = \frac{2}{3};\ \beta = \frac{3}{4};\ \gamma = 2;$
$L = 6\,\mathrm{m};\ EI = 1000\,\mathrm{kN{\cdot}m^2};$

1. Postać wyboczenia

1.1. Równanie różniczkowe osi odkształconej



I) $0 \leqslant x \leqslant \frac{2}{3}L$

$$M_y(x) = V_A\,x - \tfrac{1}{2}\,q\,x^2$$
$$EIw''(x) = \tfrac{1}{2}\,q\,x^2 - V_A\,x$$
$$EIw'(x) = \tfrac{1}{6}\,q\,x^3 - \tfrac{1}{2}\,V_A\,x^2 + C_1$$
$$EIw(x) = \tfrac{1}{24}\,q\,x^4 - \tfrac{1}{6}\,V_A\,x^3 + C_1\,x + D_1$$

II) $\frac{2}{3}L \leqslant x \leqslant L$

$$M_y(x) = V_A\,x - \tfrac{1}{2}\,q\,x^2$$
$$\tfrac{3}{4}\,EIw''(x) = \tfrac{1}{2}\,q\,x^2 - V_A\,x$$
$$\tfrac{3}{4}\,EIw'(x) = \tfrac{1}{6}\,q\,x^3 - \tfrac{1}{2}\,V_A\,x^2 + C_2$$
$$\tfrac{3}{4}\,EIw(x) = \tfrac{1}{24}\,q\,x^4 - \tfrac{1}{6}\,V_A\,x^3 + C_2\,x + D_2$$

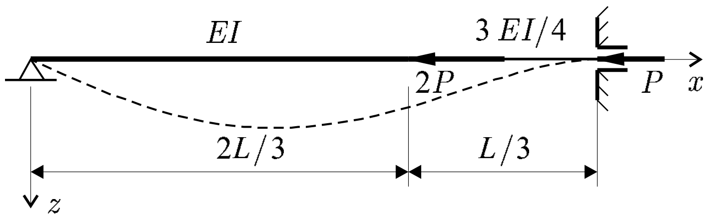
1.2. Wyznaczenie stałych z warunków brzegowych

$$\left.\begin{array}{lll}
(1) & w_I(0)=0 & \Rightarrow D_1 = 0 \\
(2) & w'_I(\tfrac{2}{3}L)=w'_{II}(\tfrac{2}{3}L) & \Rightarrow C_1 - \tfrac{4}{3}C_2 + \tfrac{2}{27}V_A L = \tfrac{4}{243}qL^3 \\
(3) & w_I(\tfrac{2}{3}L)=w_{II}(\tfrac{2}{3}L) & \Rightarrow \tfrac{2}{3}C_1 L - \tfrac{8}{9}C_2 L - \tfrac{4}{3}D_2 + \tfrac{4}{243}V_A L = \tfrac{2}{729}qL^4 \\
(4) & w'_{II}(L)=0 & \Rightarrow C_2 - \tfrac{1}{2}V_A L^2 = -\tfrac{1}{6}qL^3 \\
(5) & w_{II}(L)=0 & \Rightarrow C_2 L + D_2 - \tfrac{1}{6}V_A L^3 = -\tfrac{1}{24}qL^4
\end{array}\right\} \Rightarrow \left\{\begin{array}{l}
C_1 = \tfrac{136}{6075}qL^3 \\
C_2 = \tfrac{31}{1200}qL^3 \\
D_1 = 0 \\
D_2 = -\tfrac{1}{300}qL^4 \\
V_A = \tfrac{77}{200}qL
\end{array}\right.$$

2. Obciążenie krytyczne

2.1. Kryterium energetyczne Timoshenki

$$\int_L EI(w'')^2 \mathrm{d}x + \int_L N(w')^2 \mathrm{d}x = 0$$



$$w_I'(x) = \frac{1}{EI}(\frac{1}{6}qx^3 - \frac{77}{400}qLx^2 + \frac{136}{6075}qL^3)$$
$$w_I''(x) = \frac{1}{EI}(\frac{1}{2}qx^2 - \frac{77}{200}qLx)$$
$$N_I(x) = -3P$$

$$w_{II}'(x) = \frac{4}{3EI}(\frac{1}{6}qx^3 - \frac{77}{400}qLx^2 + \frac{31}{1200}qL^3)$$
$$w_{II}''(x) = \frac{4}{3EI}(\frac{1}{2}qx^2 - \frac{77}{200}qLx)$$
$$N_{II}(x) = -P$$

$$\int_0^{\frac{2}{3}L} EI(w_I'')^2 \mathrm{d}x + \int_{\frac{2}{3}L}^{L} \frac{3}{4}EI(w_{II}'')^2 \mathrm{d}x - P\int_0^{\frac{2}{3}L} 3(w_I')^2 \mathrm{d}x - P\int_{\frac{2}{3}L}^{L} (w_{II}')^2 \mathrm{d}x = 0$$

2.2. Wartość krytyczna siły P

$$P_{kr} = \frac{I_1 + I_2}{I_3 + I_4}$$

$$I_1 = \int_0^{\frac{2}{3}L} EI(w_I'')^2 \mathrm{d}x = \frac{q^2}{EI}\int_0^{\frac{2}{3}L}\left(\frac{1}{2}x^2 - \frac{77}{200}Lx\right)^2 \mathrm{d}x = \frac{q^2}{EI}\int_0^{\frac{2}{3}L}\left(\frac{1}{4}x^4 - \frac{77}{200}Lx^3 + \frac{77^2}{200^2}L^2x^2\right)\mathrm{d}x =$$
$$= \frac{q^2}{EI}\left(\frac{1}{20}x^5 - \frac{77}{800}Lx^4 + \frac{5929}{120000}L^2x^3\right)\Bigg|_0^{\frac{2}{3}L} = \frac{2687}{1215000}\frac{q^2L^5}{EI} = 2.21152\cdot 10^{-3}\frac{q^2L^5}{EI}$$

$$I_2 = \int_{\frac{2}{3}L}^{L} \frac{3}{4}EI(w_{II}'')^2 \mathrm{d}x = \frac{4\,q^2}{3\,EI}\int_{\frac{2}{3}L}^{L}\left(\frac{1}{2}x^2 - \frac{77}{200}Lx\right)^2 \mathrm{d}x = \frac{4\,q^2}{3\,EI}\left(\frac{1}{20}x^5 - \frac{77}{800}Lx^4 + \frac{5929}{120000}L^2x^3\right)\Bigg|_{\frac{2}{3}L}^{L} =$$
$$= \frac{9203}{7290000}\frac{q^2L^5}{EI} = 1.26241\cdot 10^{-3}\frac{q^2L^5}{EI}$$

$$I_3 = \int_0^{\frac{2}{3}L} 3(w_I')^2 \mathrm{d}x = \frac{3\,q^2}{EI^2}\int_0^{\frac{2}{3}L}\left(\frac{1}{6}x^3 - \frac{77}{400}Lx^2 + \frac{136}{6075}L^3\right)^2 \mathrm{d}x =$$

$$
\begin{aligned}
&= \frac{3\,q^2}{EI^2}\int\limits_0^{\frac{2}{3}L}\left(\frac{1}{36}x^6-\frac{77}{1200}Lx^5+\frac{5929}{160000}L^2x^4+\frac{136}{18225}L^3x^3-\frac{1309}{151875}L^4x^2+\frac{18496}{36905625}L^6\right)\mathrm{d}x=\\
&= \frac{3\,q^2}{EI^2}\left(\frac{1}{252}x^7-\frac{77}{7200}Lx^6+\frac{5929}{800000}L^2x^5+\frac{34}{18225}L^3x^4-\frac{1309}{455625}L^4x^3+\frac{18496}{36905625}L^6x\right)\Bigg|_0^{\frac{2}{3}L}=\\
&= 3.62121\cdot 10^{-4}\,\frac{q^2L^7}{EI^2}
\end{aligned}
$$

$$
\begin{aligned}
I_4 &= \int\limits_{\frac{2}{3}L}^{L}(w'_{II})^2\mathrm{d}x=\frac{16\,q^9}{EI^2}\int\limits_{\frac{2}{3}L}^{L}\left(\frac{1}{6}x^3-\frac{77}{400}Lx^2+\frac{31}{1200}L^3\right)^2\mathrm{d}x=\\
&= \frac{16\,q^2}{9\,EI^2}\int\limits_{\frac{2}{3}L}^{L}\left(\frac{1}{36}x^6-\frac{77}{1200}Lx^5+\frac{5929}{160000}L^2x^4+\frac{31}{3600}L^3x^3-\frac{2387}{240000}L^4x^2+\frac{961}{1440000}L^6\right)\mathrm{d}x=\\
&= \frac{16\,q^2}{9\,EI^2}\left(\frac{1}{252}x^7-\frac{77}{7200}Lx^6+\frac{5929}{800000}L^2x^5+\frac{31}{14400}L^3x^4-\frac{2387}{720000}L^4x^3+\frac{961}{1440000}L^6x\right)\Bigg|_{\frac{2}{3}L}^{L}=\\
&= 5.82031\cdot 10^{-5}\,\frac{q^2L^7}{EI^2}
\end{aligned}
$$

$$
P_{kr}=\frac{(2.21152\cdot 10^{-3}+1.26241\cdot 10^{-3})\dfrac{q^2L^5}{EI}}{(3.62121\cdot 10^{-4}+5.82031\cdot 10^{-5})\dfrac{q^2L^7}{EI^2}}=8.2648\frac{EI}{L^2}=8.2648\frac{1000}{6^2}=229.58\mathrm{kN}
$$

$$
\mu=\sqrt{\frac{\pi^2EI}{P_{kr}L^2}}=\sqrt{\frac{\pi^2}{8.26488}}=1.09278
$$